# EXTENDING PARTIAL LATIN CUBES

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Abstract. We prove a Ryser type theorem for Latin cubes, and prove a partial Evans type result likewise for Latin cubes. We also give an example of a very sparse incompletable Latin cube.

## 1. Introduction

Four central theorems in the theory of Latin squares are Hall's theorem on the existence of Latin squares [5], Ryser's theorem [10], Smetaniuk's theorem [11] and Evans' theorem on finite embeddability [4].

For higher dimensional Latin structures, Cruse [2] has shown that any finite partial Latin hypercube can be embedded in a finite Latin hypercube, which is an analogue of Evans' theorem for arbitrary dimension, but the bound that he obtains on the size of the Latin hypercube to embed in is probably not best possible.

For the three other theorems, no corresponding generalisations to higher dimensions are known. In fact, some natural generalisations of the aforementioned theorems do not hold. For example, an $n \times n \times k$ partial Latin cube may not be completable to a full Latin cube. Kochol [7] produced examples of incompletable $n \times n \times k$ partial Latin cubes (PLC:s) for $\frac{n}{2} < k \leq n-2$, and conjectured that any $n \times n \times (\frac{n}{2} - 1)$ partial Latin cube is completable.

However, McKay and Wanless [9] has given examples of a $5 \times 5 \times 2$ and a $6 \times 6 \times 2$ incompletable partial Latin square, thus disproving Kochol's conjecture. In general, therefore, there is no hope of completing even a PLC consisting of two complete layers to a full Latin cube.

In the present paper we will, in the spirit of Ryser's theorem, find conditions for when a $k \times \ell \times m$ PLC can be extended to a $k \times \ell \times n$ PLC, and subsequently, to a $k \times n \times n$ PLC. We can thus start with a block, extend it in one dimension, and then extend in a second dimension, but we are not yet able to extend in the third dimension.

We will also investigate the analogue of Smetaniuk's theorem in three dimensions, obtaining partial results.

## 2. Ryser's theorem revisited

In what follows, we shall make use of the simple fact that a balanced bipartite graph $B$ on $2n$ vertices has a complete matching if $\delta(B) \geq \frac{n}{2}$, and, in general, a $t$-factor if $\delta(B) \geq \frac{n}{2} + t - 1$.

A $k \times \ell \times m$ *Latin box* is a $k \times \ell \times m$ partial Latin cube where all cells are filled. The set of symbols used are $[n]$ unless stated otherwise.

**Lemma 2.1.** *Let $A$ be a $k \times \ell \times m$ Latin box. If $m + 2\ell + 2k - 4 \leq n$, then $A$ can be completed to an $k \times \ell \times n$ Latin box.*

*Proof.* $A$ consists of $k\ell$ columns of length $m$, in $k$ layers. Let $C_{i,j}$ be the $j$:th column in the $i$:th layer. Let the corresponding symbols used in these columns be $\sigma_{i,j}$.

We form for each $C_{i,j}$ a bipartite graph $G_{i,j}$ with symbols $[n] \setminus \sigma_{i,j}$ on one side and rows $m+1, \ldots, n$ on the other side of the bipartition, where an edge $(r, s)$ is present if symbol $s$ can be placed in row $r$ without creating a conflict. We complete columns $C_{1,1}, C_{1,2}, \ldots, C_{1,\ell}, C_{2,1}, \ldots, C_{k,\ell}$ in this order. Completing column $C_{i,j}$ is equivalent to finding a complete matching in $G_{i,j}$. It holds that $\delta(G_{i,j}) \geq n-m-(i-1)-(j-1)$, since we have to take into account the symbols used in $C_{1,j}, \ldots, C_{i-1,j}$ and $C_{i,1}, \ldots, C_{i,j-1}$. Since $i \leq k$, $j \leq \ell$ and $m + 2\ell + 2k - 4 \leq n$, it holds that $\delta(G_i) \geq \frac{n-m}{2}$, so we can find a matching in $G_{i,j}$, and thus complete column $C_{i,j}$ without conflicts with $C_{1,j}, \ldots, C_{i-1,j}$ and $C_{i,1}, \ldots, C_{i,j-1}$. □

**Corollary 2.2.** *Let $A$ be a $2 \times 2 \times m$ partial Latin cube. If $m \leq n-4$, then $A$ can be completed to a $2 \times 2 \times n$ partial Latin cube.*

*Proof.* Set $k = 2$, $\ell = 2$ in Lemma 2.1. □

If $m = n - 2$ the corollary doesn't hold. A concrete example of this is if the symbol sets used are $[n] \setminus \sigma_{1,1} = \{1, 2\}$, $[n] \setminus \sigma_{1,2} = \{1, 3\}$, $[n] \setminus \sigma_{2,1} = \{1, 2\}$ and $[n] \setminus \sigma_{2,2} = \{2, 3\}$. Note that the Ryser condition is satisfied in every 2-dimensional layer.

If $k = n - 3$ the result probably still holds, but we would have to choose our matchings more carefully. In general, Lemma 2.1 is most likely not best possible.

Before we look at when a $k \times \ell \times n$ Latin box extends to a $k \times n \times n$ Latin box, we give an if-and-only-if condition on when a $k \times n \times n-1$ Latin box extends to a $k \times n \times n$ Latin box, and before we go for full generality, we prove the special case $k = 2$, $\ell = 2$, in the hope that the proof idea will be more transparent.

**Proposition 2.3.** *A $k \times (n-1) \times n$ partial Latin cube is completable to a $k \times n \times n$ partial Latin cube iff for each column each symbol is used exactly $k-1$ times.*

*Proof.* Observe that the top and bottom layer of each column have $n-1$ symbols out of $n$, so the only possible sizes of their overlap are $n-1$ and $n-2$. If in any column this overlap is $n-1$, one and the same symbol would be forced in the last cell of both the top layer and the bottom layer, creating a conflict. The condition is therefore necessary.

To prove sufficiency, observe that if the condition holds, each single column can be completed in a unique way, without conflict between the two layers. Thus, there will be no conflicts between the two layers. Also, because each layer by itself is completable in a unique way (since they are both partial $(n-1) \times n$ Latin rectangles), there will be no conflicts within any of the two layers. □

**Theorem 2.4.** *For $n \geq 14$, any $2 \times 2 \times n$ Latin box can be completed to a $2 \times n \times n$ Latin box.*

*Proof.* Let $\pi_{i,j}$ be the permutation in the $j$:th row in the $i$:th layer for $i, j \in \{1, 2\}$. We shall seek to find a derangement $d$ such that $d(\pi_{2,1}(s)) \neq \pi_{1,1}(s)$, $d(\pi_{2,1}(s)) \neq \pi_{1,2}(s)$, $d(\pi_{2,2}(s)) \neq \pi_{1,1}(s)$ and $d(\pi_{2,2}(s)) \neq \pi_{1,2}(s)$ for all symbols $1 \leq s \leq n$. We will impose some further restrictions on $d$, but for now, let's suppose we've found such a $d$.

We now complete the bottom layer up until the $(n-2)$:th row. In doing so, however, we will also see to it that we avoid conflicts with $d^{-1} \circ \pi_{2,1}$ and $d^{-1} \circ \pi_{2,2}$.

For example, in the first column, we will of course have to avoid using symbols $\pi_{1,1}(1)$ and $\pi_{1,2}(1)$, but we will also avoid symbols $d^{-1}(\pi_{2,1}(1))$ and $d^{-1}(\pi_{2,2}(1))$. With these extra restrictions, since $d(\pi_{2,1}(s)) \neq \pi_{1,1}(s)$, $d(\pi_{2,1}(s)) \neq \pi_{1,2}(s)$, $d(\pi_{2,2}(s)) \neq \pi_{1,1}(s)$ and $d(\pi_{2,2}(s)) \neq \pi_{1,2}(s)$, we can complete the bottom layer except the last two rows, by Hall's theorem.

The symbols not yet used in the remaining two cells of the $s$:th column of the bottom layer are $d(\pi_{2,1}(s))$ and $d(\pi_{2,2}(s))$.

To complete the second layer, we place in row $3 \leq j \leq n-3$ the permutation $d^{-1} \circ \pi_{1,j}$. Since $d$ is a derangement, the two layers will not conflict, and since $d(\pi_{2,1}(s)) \neq \pi_{1,j}(s)$ and $d(\pi_{2,2}(s)) \neq \pi_{1,j}(s)$ for all $3 \leq j \leq n-3$ and all $s$, it also holds that $d^{-1} \circ \pi_{1,j}$ will not conflict with $\pi_{2,1}$ and $\pi_{2,2}$.

The symbols not yet used in the remaining two cells of the $s$:th column of the top layer are $d^{-1}(\pi_{1,1}(s))$ and $d^{-1}(\pi_{1,2}(s))$.

One way (which we shall stick to) of completing the last two rows is by setting $\pi_{1,n-1} = d \circ \pi_{2,1}$ and $\pi_{1,n} = d \circ \pi_{2,2}$ in the bottom layer, and $\pi_{2,n-1} = d^{-1} \circ \pi_{1,1}$ and $\pi_{2,n} = d^{-1} \circ \pi_{1,2}$. This means that we have to impose two further restrictions on $d$, namely that $d(\pi_{2,1}(s)) \neq d^{-1}(\pi_{1,1}(s))$ and $d(\pi_{2,2}(s)) \neq d^{-1}(\pi_{1,2}(s))$.

To summarize, we need a derangement $d$ that satisfies the following inequalities:

$$\begin{aligned} d(\pi_{2,1}(s)) &\neq \pi_{1,1}(s) \\ d(\pi_{2,1}(s)) &\neq \pi_{1,2}(s) \\ d(\pi_{2,2}(s)) &\neq \pi_{1,1}(s) \\ d(\pi_{2,2}(s)) &\neq \pi_{1,2}(s) \\ d(\pi_{2,1}(s)) &\neq d^{-1}(\pi_{1,1}(s)) \\ d(\pi_{2,2}(s)) &\neq d^{-1}(\pi_{1,2}(s)) \end{aligned}$$

Finding $d$ is equivalent to finding a matching in the complete bipartite graph $K_{n,n}$ with a number of edges removed. First of all, we must remove edges $(i, i)$, since $d$ must be a derangement. Next, each of the inequalities above effectively specifies a matching in $K_{n,n}$ that has to be removed. In total, we remove 7 matchings from $K_{n,n}$, yielding a graph $G$ with minimum degree at least $n-7$, so if $n \geq 14$ the minimum degree is at least $\frac{n}{2}$ and we can find such a $d$. □

Generalizing Theorem 2.4 to completing a $k \times \ell \times n$ array will work in a similar way.

**Theorem 2.5.** *For $n \geq 2(\ell^2(k-1) + \ell(k-1)^2 + 1)$, any $k \times \ell \times n$ Latin box can be completed to a $k \times n \times n$ Latin box.*

*Proof.* Let $\pi_{i,j}$ be the permutation in the $j$:th row in the $i$:th layer. For $2 \leq i \leq k$ we shall seek to find a set of derangements $d_i$ such that $d_i(\pi_{i,j_1}(s)) \neq \pi_{1,j_2}(s)$ and in general $d_{i_1}(\pi_{i_1,j_1}(s)) \neq d_{i_2}(\pi_{i_2,j_2}(s)$ for $i_1 \neq i_2$, $1 \leq j_1, j_2 \leq \ell$ and for all $s$. The condition $d_{i_1}(\pi_{i_1,j_1}(s)) \neq d_{i_2}(\pi_{i_2,j_2}(s)$ means in particular that the derangements are mutual derangements. For ease of notation in the sequel, we also set $d_1 = \text{id}$. Again, we will impose some further restrictions on the $d_i$, but for now, let's suppose we've found such a set of $d_i$.

We now complete the bottom layer up until the $(n-(k-1)\ell)$:th row. In doing so, however, we will also see to it that we avoid conflicts with $d_i^{-1} \circ \pi_{i,j}$ for all $2 \leq i \leq k$, $1 \leq j \leq \ell$. By Hall's theorem, this is possible.

To complete the first $n-(k-1)\ell$ rows of the $i$:th layer, we place in row $\ell + 1 \leq j \leq n-(k-1)\ell$ the permutation $d_i^{-1} \circ \pi_{1,j}$. Since $d_i$ is a derangement, layers 1 and $i$ will not conflict, and since $d_i(\pi_{i,j_1}(s)) \neq \pi_{1,j_2}(s)$ for all $1 \leq j_1, j_2 \leq \ell$ and all $s$, there will be no conflicts in the $i$:th layer. Also, since $d_{i_1}(s) \neq d_{i_2}(s)$ for all $s$, there will be no conflict between layers $i_1$ and $i_2$.

The symbols not yet used in the remaining $(k-1)\ell$ cells of the $s$:th column of the $i$:th layer are $d_i^{-1}(d_{i_1}(\pi_{i_1,j}(s)))$ for $i_1 \neq i$, $1 \leq j \leq \ell$, so we can complete the last $(k-1)\ell$ rows by setting $\pi_{i,n-(k-1)\ell+(i_1-1)\ell+j} = d_i^{-1} \circ d_{i_1} \circ \pi_{i_1,j}$ for $1 \leq i_1 \leq k$, $i \neq i_1$, $1 \leq j \leq \ell$ in the $i$:th layer.

This means that we have to impose a number of further restrictions on the $d_i$, namely that $d_{i_1}^{-1} \circ d_i \circ \pi_{i,j} \neq d_{i_2}^{-1} \circ d_i \circ \pi_{i,j}$ for all $i_1 \neq i_2$, and all $1 \leq j \leq \ell$.

To summarize, we need a set of $k-1$ derangements $d_i$ that satisfy the following inequalities:

$$\begin{array}{rcll} d_{i_1}(\pi_{i_1,j_1}(s)) & \neq & d_{i_2}(\pi_{i_2,j_2}(s) & \text{for } i_1 \neq i_2,\ 1 \leq j_1, j_2 \leq \ell, \text{ all } s \\ d_{i_1}^{-1} \circ d_i \circ \pi_{i,j} & \neq & d_{i_2}^{-1} \circ d_i \circ \pi_{i,j} & \text{for } i_1 \neq i_2, \text{ and all } 1 \leq j \leq \ell \end{array}$$

We find the $d_i$ in the natural order, starting with $i = 2$. Each successfully found $d_i$ then restricts the choice of the subsequent derangements. Choosing $d_i$ is equivalent to finding a perfect matching in a bipartite graph $G_i \subset K_{n,n}$. $G_i$ is $K_{n,n}$ with $\ell^2(i-1) + \ell(i-1)^2 + 1$ matchings removed. Thus, selecting $d_k$ is the hardest, and this is possible if $n \geq 2(\ell^2(k-1) + \ell(k-1)^2 + 1)$. □

It is a testament to our limited knowledge of hypergraph matchings that the above results all go to great lengths to reduce problems most naturally stated as hypergraph problems to bipartite matching problems.

## 3. A multi-dimensional Evans' conjecture

Any partial Latin square with at most $n-1$ entries is completable, as conjectured by Evans, and proven by Smetaniuk. The most natural generalization of this would be the following conjecture.

**Conjecture 3.1.** *Let $P$ be a partial $r$-dimensional Latin hypercube of order $n$. Suppose that $P$ has at most $n-1$ entries. Then $P$ is completable.*

Since the proportion of filled cells dwindles rapidly as the number of dimensions increases, it would seem most reasonable that the conjecture is, in fact, true. Perhaps we can even allow more than $n-1$ entries in $P$, provided of course that no more than $n-1$ of them occur in any 2-dimensional substructure? As the example in Figure 1 shows, $n-1$ is really best possible. Furthermore, the example easily generalizes to any order and any dimension.

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Figure 1. The first two layers of an incompletable partial Latin cube with $n$ entries

Smetaniuk's proof for the 2-dimensional case cannot be used for higher dimensions, and one major hurdle is the lack of knowledge about the structure of Latin cubes. The following lemma essentially only specifies a situation when a partial Latin cube can be embedded in a cyclic Latin cube, but obviously, there are many partial Latin cubes that are not this well-behaved.

**Lemma 3.2.** *Let $P$ be a partial Latin cube of order $n$. Suppose there exists a cyclic permutation $\sigma$ of the symbols $1, \dots, n$ such that the partial Latin square $P^*$ obtained by superimposing $\sigma^{i-1}(l_i)$ for $1 \le i \le n$ where $l_i$ is the i:th layer of $P$, is completable. Then $P$ is completable.*

*Proof.* $P^*$ coincides with all the entries already present in the bottom layer, and there are no cconflicts between $P^*$ and any symbols already present in higher layers, since $\sigma$ is cyclic. We can therefore let $P^*$ be the bottom layer. Further, let layer $i$ be given by $\sigma^{-i}(P^*$. Since $\sigma$ is cyclic, and hence the $\sigma^{-i}$ are mutual derangements, there will be no conflicts between layers, and $\sigma^{-i}(P^*$ will coincide with all entries already present in layer $i$. □

Lemma 3.2 can be used to prove Conjecture 3.1 if we know something about the distribution of the filled cells. An example of this is given in the following corollary. Also, with appropriate modifications, the lemma can be extended to arbitrary dimension.

**Corollary 3.3.** *Let $P$ be a partial Latin cube with at most $n-1$ entries, such that no two filled cells share any coordinate. Then $P$ is completable.*

*Proof.* Since no two filled cells share any coordinate, the permutation $\sigma$ from Lemma 3.2 can be found quite easily. □

A special case of Lemma 3.2 is when all entries are in one layer, in which case we complete that layer, and use *any* cyclic permutation of the symbols to complete the cube. However, even if we only have $n-1$ entries, distributed between two parallel layers, Lemma 3.2 is not enough.

We conclude this section with a result not covered by Lemma 3.2, where the entries already present lie in two perpendicular layers.

**Theorem 3.4.** *Let $P$ be a partial Latin cube all of whose at most $n-1$ entries have either first coordinate 1 or second coordinate 1. Then $P$ is completable.*

*Proof.* Let $L_1$ be the layer with first coordinate 1, and $L_2$ the layer with second coordinate 1. Further, let $S = L_1 \cap L_2$, the *spine*.

By Smetaniuk's theorem, each $L_i$ is completable separately. Suppose without loss of generality that $L_1$ has fewer entries than $L_2$, and complete $L_2$ arbitrarily. This of course adds entries to $L_1$, since $L_1$ and $L_2$ share the cells in $S$. We denote by $L^*$ the layer $L_1$ with the additional entries in $S$ amended. We shall prove that $L^*$ is completable, and that we subsequently can complete the whole cube.

$L_1$ has at most $\lfloor \frac{n-1}{2} \rfloor$ entries, so by permuting rows and columns in $L^*$, keeping $S$ in its place (though entries in $S$ may be rearranged), we can fit all the entries in $L_1 \setminus S$ in a subsquare $R$ of dimensions $\lfloor \frac{n-1}{2} \rfloor \times \lfloor \frac{n-1}{2} \rfloor$.

We can assume that $R$ occupies the first $\lfloor \frac{n-1}{2} \rfloor$ rows of $L^*$. We form $R^*$ by amending to $R$ the $\lfloor \frac{n-1}{2} \rfloor$ entries from $L^*$ occupying the first $\lfloor \frac{n-1}{2} \rfloor$ rows. We may then assume that $R^*$ fits in the first $\lfloor \frac{n-1}{2} \rfloor + 1$ columns of $L_1$. We can easily fill all the empty cells in $R^*$, and if we try to extend this to a completion of $L_1$, we find that the condition from Ryser's theorem demands that each symbol be used at least $\lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor + 1 - n \leq 0$ times, which is trivially satisfied. We still need to consider the entries in $S \setminus R^*$ that we added when completing $L_2$. They may not coincide with $L$, but if not, we just permute rows of $L$ to make it so.

We have proven that the two layers $L_2$ and $L_1$ can be completed one after the other. To extend this to the whole cube, suppose that the columns of the filled $L_1$ are the permutations $p_1, \ldots, p_k, \ldots, p_n$. To fill cell $(i, j, k)$ we use the symbol in position $s(i, 1, k) + j - 1$ of $p_k$, where $s(i, j, k)$ denotes the symbol in position $(i, j, k)$.

To see that the resulting structure is a Latin cube, observe that all $p_k$ are mutual derangements, so there will be no conflicts in the $k$-dimension. Since $L_1$ and $L_2$ have been completed to Latin squares, there will be no conflict in the $i$- or $j$-dimensions. □

## 4. Concluding remarks

In the proof of Theorem 3.4 a construction of Latin cubes from two perpendicular layers was used. This construction generalizes to arbitrary dimension. For example, two permutations $p_1$ and $p_2$ can be 'composed' to form a Latin square $L = p_1 \circ p_2$, by taking $p_1$ to be the first row and $p_2$ the first column and, if $p_2(i) = s(1, j)$, placing the symbol $p_2(i + j - 1)$ in cell $(i, j)$, where $i + j - 1$ is taken modulo $n$. The resulting Latin square will have symbols in the same cyclic order $p_2$ in each column, with the starting points being given by $p_1$. Of course, the roles of $p_1$ and $p_2$ can be interchanged.

The Latin squares that can be constructed in this way are exactly the Latin squares that are isotopically equivalent to the basic cyclic Latin square, namely with entry $i + j - 1$ modulo $n$ in cell $(i, j)$. For Latin cubes, the corresponding construction gives more than the cyclic Latin cubes, with entry $i + j + k - 2$ modulo $n$ in cell $(i, j, k)$ but still a far cry from all Latin cubes.

We would like to pose the following problem, which seems possible to solve. We conjecture that there indeed is such an $N$.

**Problem 1.** *Find an $N$ such that $n \geq N$ implies that any partial Latin cube of order $n$ consisting of two full parallel layers is always completable.*

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